

Collisional energy transfer in two-component plasmas

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Abstract: The friction in plasmas consisting of two species with different temperatures is discussed together with the consequent energy transfer. It is shown that the friction between the two species has no effect on the ion acoustic mode in a quasi-neutral plasma. Using the Poisson equation instead of the quasi-neutrality reveals the possibility for an instability driven by the collisional energy transfer. However, the different starting temperatures of the two species imply an evolving equilibrium. It is shown that the relaxation time of the equilibrium electron-ion plasma is, in fact, always shorter than the growth rate time, and the instability can thus never effectively take place. The results obtained here should contribute to the definite clarification of some contradictory results obtained in the past.

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Plasmas both in the laboratory and in space are frequently in the state of partial thermodynamic equilibrium [1]-[3] (i.e. with an initial temperature disparity of the plasma constituents). Collisions in such plasmas will after some time eventually result in equal temperatures of the species, implying an evolving plasma. However, there exists a long standing controversy in the literature, which deals with the effects of this temperature disparity on the ion acoustic (IA) waves.

In Ref. [4] it is claimed that the corresponding energy transfer may result in the instability of the acoustic mode at large wavelengths (within the quasi-neutrality limit), and that this growth may be described within the fluid theory. The *necessary* condition for the instability obtained in Ref. [4] for an electron-ion plasma is, in fact, very easily satisfied because it requires only a very small temperature difference between the two species (electrons and ions), viz. $T_e > 4T_i/3$. This instability condition is obtained by using the energy equations including the source/sink terms originating from the collisional transfer, together with the corresponding friction force terms in the momentum equations. The *sufficient* instability condition is stronger because of additional dissipative effects, like viscosity and thermal conductivity.

However, the current-less instability described in Ref. [4] is based on a model which disregards the same temperature disparity in the description of the equilibrium, which, due to the same reasons, must be time evolving. In other words, the effects of collisions in the equilibrium have been explicitly neglected. These effects have been discussed in Ref. [5], published one year after Ref. [4] and for the same quasi-neutrality case. There, it is claimed that there is no instability for any temperature ratio of the two plasma components, and moreover, that this holds even in a current-carrying plasma, as long as the difference between the electron and ion equilibrium velocity remains below the sound speed. All that was needed to come to that conclusion was to let the equilibrium plasma evolve freely in the presence of the given temperature difference. However, we observe that Ref. [5] has apparently remained almost unnoticed by researchers, in contrast to the widely cited Ref. [4], see e.g. Refs. [6]-[10] and many others.

In the present work, this controversy is revisited for any two-component plasma. Essential for the problem is the energy equation describing the temperature variation. In the simplified form that we shall use, it contains only the collisional energy transfer source/sink term on the right-hand side. This simplified form is used for clarity only because, according to Ref. [4], in the absence of currents, that term alone is supposed

to yield an instability. In view of the controversy mentioned above, here we give some details following Braginskii [11], where the energy equation for any species a is given in the form:

$$\frac{3}{2}n_a\frac{\partial T_a}{\partial t} + n_aT_a\nabla \cdot \vec{v}_a + \frac{3}{2}n_a(\vec{v}_a \cdot \nabla)T_a = Q_a. \quad (1)$$

The corresponding equation for the species b has the same shape, but with a minus sign on the right-hand side. We use the Landau formula for the energy transfer source/sink term [12], $Q_a = 3m_b\nu_{ba}n_b(T_b - T_a)/m_a$, where [13]

$$\nu_{ba} = 4 \left(\frac{2\pi}{m_b} \right)^{1/2} \left(\frac{q_a q_b}{4\pi\epsilon_0} \right)^2 \frac{n_a L_{ba}}{3(T_b + T_a m_b/m_a)^{3/2}}. \quad (2)$$

The Coulomb logarithm is given by $L_{ba} = \log[r_d/b_0]$, $r_d = r_{da}r_{db}/(r_{da}^2 + r_{db}^2)^{1/2}$, $r_{dj} = v_{Tj}/\omega_{pj}$, and $b_0 = [|q_a q_b|/(4\pi\epsilon_0)]/[3(T_a + T_b)]$ is the impact parameter.

The additional source/sink term of the form $\vec{F}_{fa} \cdot (\vec{v}_a - \vec{v}_b)$, where \vec{F}_{fa} is the friction force acting on the species a , in the absence of equilibrium currents/drifts, is in fact nonlinear and will not be discussed here. The other (sink) terms, due to viscosity and thermal conductivity, are omitted only for the sake of clarity, i.e. in order to demonstrate more clearly the effect of the disputed collisional energy transfer term. The effect of these omitted terms is easily predictable. Equation (1) is valid for any species a, b , thus including the electron-ion plasma from Refs. [4]-[9].

a) In Ref. [4] the collisions in the equilibrium were explicitly ignored. In that case, the two energy equations without the equilibrium effects, corresponding to the model from Ref. [4] read:

$$\begin{aligned} \frac{\partial T_{(a,b)1}}{\partial t} + \frac{2}{3}T_{(a,b)0}\nabla \cdot \vec{v}_{(a,b)1} = & \pm 2\frac{m_b}{m_a}\nu_{ba}(T_{b1} - T_{a1}) \\ & \pm 2\nu_{ba}\frac{m_b}{m_a}(T_{b0} - T_{a0})\frac{n_{b1}}{n_0}. \end{aligned} \quad (3)$$

Here, the minus sign applies to the species b .

The two momentum equations and the two continuity equations have standard forms and there is no need to write them down here. We stress only the presence of the friction force terms in the momentum equations. These are of the form $\vec{F}_{fa} = -m_a n_a \nu_{ab}(\vec{v}_a - \vec{v}_b)$ and $\vec{F}_{fb} = -m_b n_b \nu_{ba}(\vec{v}_b - \vec{v}_a)$, respectively.

In the case of quasi-neutral perturbations, the two number densities $n_{(a,b)1}$ are calculated from the continuity equations and are made equal assuming a quasi-neutral plasma, like in Refs. [4], [5] (this is typically done when dealing with wavelengths that are much

longer than the Debye length). The dispersion equation reads:

$$\left(\omega + \frac{i4m_b\nu_{ba}}{m_a}\right) \left(\omega^2 - \frac{5}{3}k^2 \frac{T_{a0} + T_{b0}}{m_a + m_b}\right) = 0. \quad (4)$$

Hence, even using the same model as in Ref. [4], we conclude that there is neither an instability nor damping of the acoustic mode, regardless of the ratio T_{a0}/T_{b0} .

Note that the momentum conservation condition $\nu_{ab} = m_b n_b \nu_{ba} / (m_a n_a)$ is *nowhere* used in the derivation of Eq. (4). This is because the friction terms vanish in any case. In fact, from the two continuity equations we have the velocities $v_{j1} = \omega n_{j1} / (k n_0)$. These expressions, together with the assumption of quasi-neutrality, cancel the friction completely. This remains so for any two species a and b as long as their charge numbers Z_a and Z_b are constant. Further in the text we assume singly charged species.

b) The derivations are now repeated for isothermal quasi-neutral perturbations. In addition, the energy equation may be omitted in the equilibrium also assuming that the relaxation time for the equilibrium temperature is much longer than the period of wave oscillations. Keeping the full friction force \vec{F}_f in both momentum equations, and within the same quasi-neutrality limit, yields a *real* dispersion equation $\omega^2 = k^2(T_{a0} + T_{b0}) / (m_a + m_b)$. In the given limit the collisions (through friction) do not affect the isothermal ion acoustic mode. This fact is usually overlooked in the literature. The collisions appear in Eq. (4) only from the energy equations, yet they do not affect the IA mode.

b.1) Using the Poisson equation instead of quasi-neutrality, for isothermal perturbations we obtain coupled and damped IA and Langmuir waves

$$\begin{aligned} \omega^4 + i(\nu_{ab} + \nu_{ba})\omega^3 - \left[k^2(v_{Ta}^2 + v_{Tb}^2) + \omega_{pa}^2 + \omega_{pb}^2\right]\omega^2 \\ - ik^2(\nu_{ab}v_{Tb}^2 + \nu_{ba}v_{Ta}^2)\omega \\ + k^4v_{Ta}^2v_{Tb}^2 + k^2(v_{Ta}^2\omega_{pb}^2 + v_{Tb}^2\omega_{pa}^2) = 0. \end{aligned} \quad (5)$$

In the collision-less limit the two modes (5) decouple by setting $T_a = T_b$, though strictly speaking this has not much sense because in this case the acoustic mode may lose its electrostatic nature, especially in pair-plasmas. For a pair (pair-ion, electron-positron) collision-less plasma the solutions are $\omega^2 = \omega_p^2 + k^2(v_{Ta}^2 + v_{Tb}^2)/2 \pm [\omega_p^4 + k^4(v_{Ta}^2 - v_{Tb}^2)/4]^{1/2}$.

In the low frequency limit $\omega \ll \omega_{p(a,b)}$ and for an e-i plasma, from Eq. (5) we have $\omega^2 = k^2v_s^2 - i2\nu_{ei}\omega m_e r_{de}^2 k^2 / m_i$, so that the IA mode is damped

$$\omega = \pm k v_s \left(1 - r_{de}^2 k^2 \frac{\nu_{ie} r_{de}^2}{v_s^2}\right)^{1/2} - i \nu_{ie} r_{de}^2 k^2. \quad (6)$$

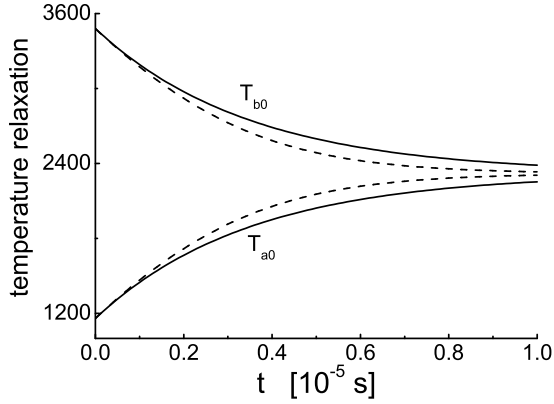


Figure 1: Approximative (full lines), and exact relaxation with time-dependent collision frequencies (dashed lines) of equilibrium temperatures (7).

We have used $\nu_{ie} = m_e \nu_{ei} / m_i$, and $v_s^2 = c_s^2 + v_{Ti}^2$. The damping in Eq. (6) is k -dependent.

c) From Eq. (1) it is seen that in a quasi-neutral homogeneous equilibrium, without flows/currents, *the equilibrium temperature is also evolving in time* as

$$\frac{\partial T_{(a,b)0}}{\partial t} = \pm 2 \frac{m_b}{m_a} \nu_{ba} (T_{b0} - T_{a0}). \quad (7)$$

Keeping the collision frequencies constant this gives the two temperatures $T_{(a,b)0} = [\hat{T}_{(a,b)0}(1 + \exp(-4\nu_{ab}t)) + \hat{T}_{(a,b)0}(1 - \exp(-4\nu_{ab}t))]/2$ evolving towards the common value $(\hat{T}_{a0} + \hat{T}_{b0})/2$. On the other hand, solving (7) numerically with time dependent collision frequencies (2) gives a slightly faster relaxation for the two temperatures. To get a feeling on the time scale, this is presented in Fig. 1 by taking $n_0 = 10^{18} \text{ m}^{-3}$ and $\hat{T}_{a0} = 0.1 \text{ eV}$, $\hat{T}_{b0} = 3\hat{T}_{a0}$.

Eq. (7) is to be used in the linearization of Eq. (1), which in the case $n_{a0} = n_{b0} = n_0$ yields:

$$\begin{aligned} \frac{\partial T_{a1}}{\partial t} + \frac{2}{3} T_{a0} \nabla \cdot \vec{v}_{a1} &= +2 \frac{m_b}{m_a} \nu_{ba} (T_{b1} - T_{a1}) \\ &- 2 \nu_{ba} \frac{m_b}{m_a} \frac{n_{a1} - n_{b1}}{n_0} (T_{b0} - T_{a0}). \end{aligned} \quad (8)$$

The corresponding equation for the component b is

$$\frac{\partial T_{b1}}{\partial t} + \frac{2}{3} T_{b0} \nabla \cdot \vec{v}_{b1} = -2 \frac{m_b}{m_a} \nu_{ba} (T_{b1} - T_{a1}). \quad (9)$$

Here, in the process of linearization yielding Eq. (9), the term $(3/2)n_{b1}\partial T_{b0}/\partial t$ on the left-hand side, cancels out with the term $-(m_b/m_a)\nu_{ba}(T_{b0} - T_{a0})n_{b1}$ on the right-hand side after using the equilibrium equation (7) for the species b .

Hence, both Eqs. (8) and (9) are obtained taking into account the evolution of the equilibrium. There appears an additional asymmetry between the two energy equations (apart from the opposite signs of the first term on the right-hand side), due to the last term in Eq. (8). This extra asymmetry is a consequence of the fact that the internal energy of the two species may also change due to the presence of the new ingredient in the system, i.e., the perturbed electric field $n_{a1} - n_{b1} = \varepsilon_0 \nabla \cdot \vec{E}_1 / e$ (in the presence of the necessary collisions of course). However, it vanishes if the quasi-neutrality condition is used on the right-hand side in Eq. (8), which sometimes may be permissible in higher order terms but not in general, for example assuming that the source/sink term in the energy equations gives only small imaginary corrections to the frequency.

However, regardless of the fact that the last term in Eq. (8) is used or not, the effects of the evolving equilibrium remain within Eq. (8) in both cases. Note also that the cancelation of the terms in the equation for the species b (which is due to evolving equilibrium as described above) remains intact.

c.1) We stress that Eq. (4) is obtained also by using Eqs. (8, 9) in the quasi-neutral limit (implying that the last term in Eq. (8) is omitted). Hence, the IA mode appears unaffected by friction in the quasi-neutral limit even if the energy equations are used and the equilibrium is described correctly as evolving.

c.2) We now use the two energy equations (8, 9) *with the Poisson equation*. The dispersion equation becomes

$$\begin{aligned}
& \omega^6 + i\nu_{ba} \left(1 + \frac{5m_b}{m_a}\right) \omega^5 - \omega^4 \left[\frac{5}{3} k^2 (v_{Ta}^2 + v_{Tb}^2) + \omega_{pa}^2 \right. \\
& \quad \left. + \omega_{pb}^2 + 4\nu_{ba}^2 \frac{m_b}{m_a} \left(1 + \frac{m_b}{m_a}\right) \right] + i\omega^3 \nu_{ba} \left[k^2 v_{Tb}^2 \frac{2m_b^2}{m_a^2} \right. \\
& \quad \left. - k^2 v_{Ta}^2 \left(\frac{5}{3} + \frac{22m_b}{3m_a} \right) - 4\omega_{pa}^2 \left(1 + \frac{m_b}{m_a}\right) - 7k^2 v_{Tb}^2 \frac{m_b}{m_a} \right] \\
& \quad + \omega^2 \left[\frac{25}{9} k^4 v_{Ta}^2 v_{Tb}^2 + \frac{5k^2}{3} (v_{Ta}^2 \omega_{pb}^2 + v_{Tb}^2 \omega_{pa}^2) \right. \\
& \quad \left. + 4k^2 \nu_{ba}^2 \frac{m_b}{m_a} \left(v_{Ta}^2 \left(\frac{2}{3} + \frac{m_b}{m_a} \right) + v_{Tb}^2 \frac{m_b^2}{m_a^2} \left(\frac{8}{3} - \frac{m_b}{m_a} \right) \right) \right] \\
& \quad + i10\omega \nu_{ba} k^2 \left[k^2 v_{Tb}^2 \frac{m_b}{m_a} \left(v_{Ta}^2 - \frac{v_{Tb}^2 m_b}{3 m_a} \right) + \frac{2}{3} \omega_{pa}^2 (v_{Ta}^2 \right. \\
& \quad \left. + v_{Tb}^2 \frac{m_b}{m_a}) \right] + 4k^4 \nu_{ba}^2 \frac{m_b}{m_a} \left(v_{Ta}^2 - v_{Tb}^2 \frac{m_b}{m_a} \right)^2 = 0.
\end{aligned} \tag{10}$$

Solving for the IA mode yields approximately the frequency of the IA mode

$$\omega_{IA}^2 = \frac{5k^2(\omega_{pb}^2 v_{Ta}^2 + \omega_{pa}^2 v_{Tb}^2 + 5k^2 v_{Ta}^2 v_{Tb}^2/3)}{3 \left[\omega_{pa}^2 + \omega_{pb}^2 + (5/3)k^2(v_{Ta}^2 + v_{Tb}^2) \right]}. \quad (11)$$

The growth rate is:

$$\begin{aligned} \gamma = & \frac{1}{2 \left[\omega_{pa}^2 + \omega_{pb}^2 + (5/3)k^2(v_{Ta}^2 + v_{Tb}^2) - 2\omega_{IA}^2 \right]} \times \\ & \times \left\{ \nu_{ab} \left(\omega_{IA}^2 - (5/3)k^2 v_{Tb}^2 \right) \left(1 + (4/3)k^2 v_{Ta}^2 / \omega_{IA}^2 \right) \right. \\ & + \nu_{ba} \left(\omega_{IA}^2 - (5/3)k^2 v_{Ta}^2 \right) \left[1 + (4/3)\nu_{ab}k^2 v_{Tb}^2 / (\nu_{ba}\omega_{IA}^2) \right] \\ & + 2(1 - T_{a0}/T_{b0}) \left(\nu_{ab}k^2 c_s^2 / \omega_{IA}^2 \right) \left[\omega_{IA}^2 - (5/3)k^2 v_{Tb}^2 \right. \\ & \quad \left. - (8\nu_{ab}\nu_{ba} / \omega_{IA}^2) \left(\omega_{IA}^2 - k^2 v_{Ta}^2 \right) \right. \\ & \quad \left. \left. + (8\nu_{ab}^2 / \omega_{IA}^2) \left(\omega_{IA}^2 - k^2 v_{Tb}^2 \right) \right] \right\}. \end{aligned} \quad (12)$$

In principle, Eq. (12) reveals the possibility for a growing IA mode if the Poisson equation is used instead of the quasi-neutrality in a time-evolving plasma. For example, this can be easily demonstrated in the limit of negligible terms originating from the last term in Eq. (8), i.e., on condition $|(T_{a1} - T_{b1})/(n_{a1} - n_{b1})| \gg |T_{a0} - T_{b0}|/n_0$, or in an alternative form, $|(T_{a1} - T_{b1})/(T_{a0} - T_{b0})| \gg r_{db}^2 k^2 |q_b \phi_1|/T_{b0}$. In that limit, the numerical solution of Eq. (10) yields the growth-rate of the IA mode in an electron-ion plasma that is presented in Fig. 2. Here, $n_0 = 10^{18} \text{ m}^{-3}$ and we take several values of T_e/T_i , where $T_i = 0.1 \text{ eV}$. The growth rate increases with T_e/T_i but only up to $T_e/T_i \simeq 3$. For even higher values of the temperature ratio the instability ceases, this is represented by the dashed ($T_e/T_i \simeq 10$) line.

However, we stress that the system evolves in time, and in order to have a reasonable fast growth of the perturbations, the following condition must be satisfied [cf. Eq. (7)] :

$$\gamma_r \equiv 2(m_b/m_a)\nu_{ba} \ll \gamma. \quad (13)$$

Taking the electron-ion case like in Ref. [4] and the corresponding self-evident conditions $m_b \ll m_a$, $T_{a0} < T_{b0}$, $k^2 v_{Ta}^2 < \omega_{IA}^2 < \omega_{pa}^2 < \omega_{pb}^2$, from Eq. (12) to the leading order terms we obtain

$$\gamma - \gamma_r \simeq -\frac{\nu_{ba}}{2 \left(\omega_{pb}^2 + 5k^2 v_{Tb}^2/3 \right)} \left\{ 4\omega_{pa}^2 - \omega_{IA}^2 \right\}$$

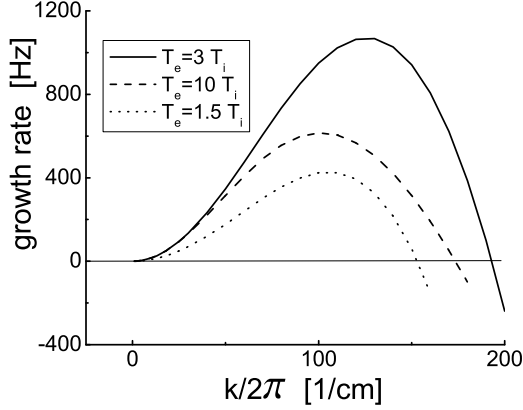


Figure 2: The growth rate of the IA mode in electron-proton plasma with $n_0 = 10^{18} \text{ m}^{-3}$ and for several values of T_e/T_i .

$$+ \frac{2k^2 c_s^2}{\omega_{IA}^2} \left[k^2 v_{Ta}^2 \left(\frac{5}{3} + \frac{8\nu_{ab}^2}{\omega_{IA}^2} \right) + 8\nu_{ab}^2 \right] \Big\}. \quad (14)$$

Hence, because always $\omega_{pa} \geq \omega_{IA}$, here we have

$$\gamma < \gamma_r, \quad (15)$$

i.e., the system relaxes on a time scale that is (much) shorter than the eventual growth time, and consequently the assumed instability actually can not develop. We note that this is in agreement with some experiments, e.g. in a Q-machine plasma [14] where the instability has never been observed even by cooling the ions to near room temperature while keeping various temperatures for the electrons.

To summarize, the long existing controversy dealing with the stability of the ion acoustic mode in plasmas in the state of partial thermodynamic equilibrium has been revisited. The results obtained here can be summarized as follows: i) The friction does not affect the IA mode in a quasi-neutral plasma; ii) Even using the non-evolving model equivalent to Ref. [4], there is no instability of the IA mode, contrary to claims from Ref. [4]; iii) When the equilibrium plasma is properly described as evolving in time, and as long as the quasi-neutrality is used, collisions do not produce a growth of the ion acoustic mode; iv) When the Poisson equation is used instead of quasi-neutrality, in principle there is a possibility for a positive growth-rate of the IA mode. It appears as a combined effect of the breakdown of the charge neutrality from one side (introduced by the Poisson equation), and the heat transfer (the compressibility and advection in energy equation) from the other side, all within the background of a time-evolving plasma. However, as

the equilibrium plasma evolves in time, with the relaxation time τ_r given in Eq. (7), the obtained growth time must be (much) shorter than the relaxation time. Yet, this shows to be impossible and we conclude that there is no instability in the electron-ion plasma with an initial temperature disparity.

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